Takahashi's, Fan-Browder's and Schauder-Tychonoff's fixed point theorems in a vector lattice (Nonlinear Analysis and Convex Analysis)

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Abstract

The purpose of this paper is to show fixed point theorems using the topology introduced by [2]. In particular, we obtain Takahashi's fixed point theorem in the case where the whole space is a vector lattice with unit. Using Takahashi's fixed point theorem in this space, we also obtain Fan-Browder's fixed point theorem and Schauder-Tychonoff's fixed point theorem.

1 Introduction

There are many fixed point theorems in a topological vector space, for instance, Takahashi's fixed point theorem and Fan-Browder's fixed point theorem in a topological vector space, Tychonoff's fixed point theorem in a locally convex space, Schauder's fixed point theorem in a normed space, and so on; see for example [7].

Takahashi [6] proved the following; see also [7].

Takahashi's fixed point theorem. *Let X be a Hausdorff topological vector space, Y a compact subset of X and Z a convex subset of Y. Suppose that f a mapping from Z into 2^Y satisfies*

(0) $f^{-1}(y)$ *is convex for any* $y \in Y$,

and there exists a mapping g from Z into 2^Y satisfying the following conditions:

(1) $g(z)$ *is a subset of* $f(z)$ *for any* $z \in Z$;

(2) $g^{-1}(y)$ *is non-empty for any* $y \in Y$;

(3) $g(z)$ *is an open subset of* X *for any* $z \in Z$.

Then there exists $z_0 \in Z$ *such that* $z_0 \in f(z_0)$.

In the mentioned above, $f^{-1}(y) = \{x \mid y \in f(x)\}$.

In this paper, we consider fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum $\vee$ and the infimum $\wedge$, and also an order is introduced from these operators; see also [5, 8] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [2] one method is introduced in the case of the vector lattice with unit.

The purpose of this paper is to show fixed point theorems using the topology introduced by [2]. In particular, we obtain Takahashi's fixed point theorem in the case where X is a vector lattice with unit. Using Takahashi's fixed point theorem in this space, we also obtain Fan-Browder's fixed point theorem and Schauder-Tychonoff's fixed point theorem.

2 Topology in a vector lattice

In this section we introduce a topology in a vector lattice introduced by [2].

Let X be a vector lattice. $e \in X$ is said to be an unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of X. In the case where X is the set of real numbers $\mathbf{R}$, $\mathcal{K}_\mathbf{R}$ is the set of positive real numbers. Let X be a vector lattice with unit and let Y be a subset of X. Y is said to be open if for any $x \in Y$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_\mathbf{R}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset Y$. Let $\mathcal{O}_X$ be the class of open subsets of X. Y is closed if $Y^C \in \mathcal{O}_X$. For $e \in \mathcal{K}_X$ and for an interval $[a, b]$ we consider the following subset

$$[a, b]^e = \{x \mid \text{ there exists some } \varepsilon \in \mathcal{K}_\mathbf{R} \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$

By the definition of $[a, b]^e$ it is easy to see that $[a, b]^e \subset [a, b]$. A mapping from $X \times \mathcal{K}_X$ into $(0, \infty)$ is said to be a gauge. Let Δ_X be the class of gauges in X. For $x \in X$ and $\delta \in \Delta_X$, $O(x, \delta)$ is defined by

$$O(x, \delta) = \bigcup_{e \in \mathcal{K}_X} [x - \delta(x, e)e, x + \delta(x, e)e]^e.$$

$O(x, \delta)$ is said to be a δ-neighborhood of x. Suppose that for any $x \in X$ and for any $\delta \in \Delta_X$ there exists $U \in \mathcal{O}_X$ such that $x \in U \subset O(x, \delta)$.

Lemma 1. *Let X be a vector lattice with unit and Y a subset of X. Then the following are equivalent.*

(1) *Y is an open subset of X.*

(2) *There exists $\delta \in \Delta_X$ such that $O(x, \delta)$ is a subset of Y for any $x \in Y$.*

(3) *For any $x \in Y$ there exists $\delta \in \Delta_X$ such that $O(x, \delta)$ is a subset of Y.*

Proof. We first show that (1) implies (2). Suppose that $Y \in \mathcal{O}_X$. Let $x \in Y$ and $e \in \mathcal{K}_X$. Since $Y \in \mathcal{O}_X$, there exists a positive number $\delta(x, e)$ such that $[x - \delta(x, e)e, x + \delta(x, e)e] \subset Y$. Then $\delta \in \Delta_X$. Let $y \in O(x, \delta)$ arbitrary. Then there exists $e \in \mathcal{K}_X$ such that $y \in [x - \delta(x, e)e, x + \delta(x, e)e]^e$. Then it follows that

$$y \in [x - \delta(x, e)e, x + \delta(x, e)e]^e \subset [x - \delta(x, e)e, x + \delta(x, e)e] \subset Y.$$

Therefore $O(x, \delta) \subset Y$. It is obvious that (2) implies (3). So next we show that (3) implies (1). Suppose that for any $x \in Y$ there exists $\delta \in \Delta_X$ such that $O(x, \delta) \subset Y$. For any $e \in \mathcal{K}_X$ let $\delta < \delta(x, e)$. Then $[x - \delta e, x + \delta e] \subset [x - \delta(x, e)e, x + \delta(x, e)e]^e$. By the definition of $O(x, \delta)$, we have

$$[x - \delta e, x + \delta e] \subset [x - \delta(x, e)e, x + \delta(x, e)e]^e \subset O(x, \delta) \subset Y.$$

Therefore $Y \in \mathcal{O}_X$. □

For a subset Y of X we denote by $cl(Y)$ and $int(Y)$, the closure and the interior of Y, respectively. Let X and Y be vector lattices with unit, $x_0 \in Z \subset X$ and f a mapping from Z into Y. f is said to be continuous in the sense of topology at x_0 if for any $V \in \mathcal{O}_Y$ with $f(x_0) \in V$ there exists $U \in \mathcal{O}_X$ with $x_0 \in U$ such that $f(U \cap Z) \subset V$.

3 Takahashi's and Fan-Browder's fixed point theorems

In this section we show Takahashi's fixed point theorem and Fan-Browder's fixed point theorem using the topology introduced in Section 2.

Let X be a vector lattice with unit. X is said to be Hausdorff if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $O_1, O_2 \in \mathcal{O}_X$ such that $x_1 \in O_1$, $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. A subset Y of X is said to be compact if for any open covering of Y there exists a finite sub-covering. A subset Y of X is said to be normal if for any closed subsets F_1 and F_2 with $F_1 \cap F_2 \cap Y = \emptyset$ there exists $O_1, O_2 \in \mathcal{O}_X$ such that $F_1 \subset O_1$, $F_2 \subset O_2$ and $O_1 \cap O_2 \cap Y = \emptyset$. Moreover the following hold.

(1) Let X be a Hausdorff vector lattice with unit and Y a compact subset of X. Then Y is normal.

(2) Let X be a vector lattice with unit and Y a normal and closed subset of X. If $Y \subset \bigcup_{i=1}^n O_i$, where $O_i \in \mathcal{O}_X$, then there exists a continuous function β_i in the sense of topology from Y into $[0,1]$ for each i such that $\beta_i(y) = 0$ for any $y \in O_i^C \cap Y$ and $\sum_{i=1}^n \beta_i(y) = 1$.

A vector lattice is said to be Archimedean if it holds that $x = 0$ whenever there exists $y \in X$ with $y \geq 0$ such that $0 \leq rx \leq y$ for any $r \in \mathcal{K}_{\mathbf{R}}$. A mapping N from $X \times \mathcal{K}_X$ to $[0, \infty]$ is defined by $N(x, e) = \sup\{r \mid r|x| \leq e\}$. Moreover we consider the following condition:

(UA) For any $e \in \mathcal{K}_X$ and for any $\{x_1, \cdots, x_m\}$ which is a linearly independent subset of X there exists $M \in \mathcal{K}_{\mathbf{R}}$ such that $N\left(\sum_{i=1}^m k_i x_i, e\right) \leq M$ for any $k_1, \cdots, k_m \in \mathbf{R}$ with $\sum_{i=1}^m k_i^2 = 1$.

Lemma 2. *Every Archimedean vector lattice satisfies the condition* (UA).

Proof. By [8, Theorem IV.11.1] for any Archimedean vector lattice X there exists the completion $\hat{X}$ of X. By [8, Theorem V.4.2] for the complete vector lattice $\hat{X}$ there exists an extremally disconnected compact set Ω and a vector sublattice Y of $C_\infty(\Omega)$ such that $\hat{X}$ is isomorphic to Y, where

$$C_\infty(\Omega) = \left\{ f \;\middle|\; \begin{array}{l} f \text{ is continuous from } \Omega \text{ into } [-\infty, \infty] \text{ and} \\ f^{-1}(\{\pm\infty\}) \text{ is nowhere dense} \end{array} \right\}.$$

Therefore it may be assumed that X is a vector sublattice of $C_\infty(\Omega)$. Then

$$\begin{aligned} N\left(\sum_{i=1}^m k_i x_i, e\right) &= \sup\left\{ r \;\middle|\; r\left|\sum_{i=1}^m k_i x_i(\omega)\right| \leq e(\omega) \text{ for any } \omega \in \Omega \right\} \\ &= \inf\left\{ \frac{e(\omega)}{\left|\sum_{i=1}^m k_i x_i(\omega)\right|} \;\middle|\; \omega \in \Omega \right\}. \end{aligned}$$

Let $S = \left\{(k_1, \cdots, k_m) \mid \sum_{i=1}^m k_i^2 = 1\right\}$ and E_ω a mapping from S into $[0, \infty]$ defined by

$$E_\omega(k_1, \cdots, k_m) = \frac{e(\omega)}{\left|\sum_{i=1}^m k_i x_i(\omega)\right|}.$$

Then for any $(k_1, \cdots, k_m) \in S$ there exists $\omega \in \Omega$ such that $e(\omega) \neq \infty$ and $\sum_{i=1}^m k_i x_i(\omega) \neq 0$. Actually assume that there exists $(k_1, \cdots, k_m) \in S$ such that $e(\omega) = \infty$ or $\sum_{i=1}^m k_i x_i(\omega) = 0$ for any $\omega \in \Omega$. Let $\Omega' = \{\omega \mid \sum_{i=1}^m k_i x_i(\omega) \neq 0\}$. Since each x_i is continuous, Ω' is open. On the other hand, since $\Omega' \subset \{\omega \mid e(\omega) = \infty\}$, Ω' is nowhere dense. It is a contradiction. Therefore for any $(k_1, \cdots, k_m) \in S$ there exists $\omega \in \Omega$ such that $e(\omega) \neq \infty$ and $\sum_{i=1}^m k_i x_i(\omega) \neq 0$. Let

$$T_\omega = \left\{ (k_1, \cdots, k_m) \;\middle|\; (k_1, \cdots, k_m) \in S, \sum_{i=1}^m k_i x_i(\omega) \neq 0 \right\}.$$

Then $\bigcup_{\omega\in\{\omega|e(\omega)\neq\infty\}} T_\omega = S$. Since S is compact and each T_ω is open, there exists $\omega_1,\cdots,\omega_p \in \{\omega \mid e(\omega)\neq\infty\}$ such that $\bigcup_{j=1}^p T_{\omega_j} = S$. Let

$$E(k_1,\cdots,k_m) = \min\{E_{\omega_j}(k_1,\cdots,k_m) \mid j=1,\cdots,p\}.$$

Then E is continuous on S. Let $M = \max\{E(k_1,\cdots,k_m) \mid (k_1,\cdots,k_m)\in S\}$. Then

$$\begin{aligned} N\left(\sum_{i=1}^m k_i x_i, e\right) &= \inf\left\{ \frac{e(\omega)}{|\sum_{i=1}^m k_i x_i(\omega)|} \;\middle|\; \omega\in\Omega\right\} \\ &\leq E(k_1,\cdots,k_m) \leq M. \end{aligned}$$

Therefore X satisfies the condition (UA). □

To prove our main result, we need the following lemma.

Lemma 3. *Let X be an Archimedean vector lattice with unit and $\{x_1,\cdots,x_n\}$ a subset of X. Then $co\{x_1,\cdots,x_n\}$ is homeomorphic to a compact and convex subset of $\mathbf{R}^n$.*

Proof. Suppose that $\{x_1,\cdots,x_m\}$ is a linearly independent subset of $\{x_1,\cdots,x_n\}$ and $x_j = \sum_{i=1}^m a_{j,i}x_i$ for $j=m+1,\cdots,n$. Let $X_0 = Span\{x_1,\cdots,x_m\}$, $e_i = (0,\cdots,0,\overset{i}{1},0,\cdots,0)\in\mathbf{R}^m$ for any $i=1,2,\cdots,m$ and f a mapping from X_0 into $\mathbf{R}^m$ defined by $f\left(\sum_{i=1}^m c_i x_i\right) = \sum_{i=1}^m c_i e_i$. Then f is bijective clearly.

Since by Lemma 2 X satisfies the condition (UA), for any $e\in\mathcal{K}_X$ there exists $M\in\mathcal{K}_{\mathbf{R}}$ such that $|k_i|\leq M$ for any i if $|\sum_{i=1}^m k_i x_i|\leq e$. Actually it is shown as follows. It may be assumed that $\sum_{i=1}^m k_i^2\neq 0$. Let $e\in\mathcal{K}_X$. Since X satisfies the condition (UA), there exists $M\in\mathcal{K}_{\mathbf{R}}$ such that $N\left(\sum_{i=1}^m \frac{k_i}{\sqrt{\sum_{i=1}^m k_i^2}}x_i, e\right)\leq M$. Since

$$\sqrt{\sum_{i=1}^m k_i^2}\left|\sum_{i=1}^m \frac{k_i}{\sqrt{\sum_{i=1}^m k_i^2}}x_i\right| = \left|\sum_{i=1}^m k_i x_i\right| \leq e,$$

by the definition of N

$$|k_i| \leq \sqrt{\sum_{i=1}^m k_i^2} \leq N\left(\sum_{i=1}^m \frac{k_i}{\sqrt{\sum_{i=1}^m k_i^2}}x_i, e\right) \leq M$$

for any i. Take $\varepsilon\in\mathcal{K}_{\mathbf{R}}$ arbitrary and let $V_\varepsilon = (c_1-\varepsilon, c_1+\varepsilon)\times\cdots\times(c_m-\varepsilon, c_m+\varepsilon)$. Take $\delta\in\Delta_X$ satisfying $\delta\left(\sum_{i=1}^m c_i x_i, e\right)\leq\frac{\varepsilon}{M}$. If $\sum_{i=1}^m (c_i+h_i)x_i \in O\left(\sum_{i=1}^m c_i x_i, \delta\right)$, then $|h_i|<\varepsilon$ for any i. Therefore

$$f\left(\sum_{i=1}^m (c_i+h_i)x_i\right) = \sum_{i=1}^m (c_i+h_i)e_i \in V_\varepsilon.$$

Let $U = int\left(O\left(\sum_{i=1}^m c_i x_i, \delta\right)\right)$. Then $f(U\cap X_0)\subset V_\varepsilon$ proving that f is continuous in the sense of topology.

Conversely f^{-1} is continuous in the sense of topology. In fact, take $U\in\mathcal{O}_X$ arbitrary. By Lemma 1 there exists $\delta\in\Delta_X$ such that $O\left(\sum_{i=1}^m c_i x_i, \delta\right)\subset U$. Take $e\geq\sum_{i=1}^m |x_i|$ and $\varepsilon\in\mathcal{K}_{\mathbf{R}}$ with $\varepsilon\leq\delta\left(\sum_{i=1}^m c_i x_i, e\right)$. If $\sum_{i=1}^m (c_i+h_i)e_i\in V_\varepsilon$, then $|\sum_{i=1}^m h_i x_i| < \varepsilon e$. Therefore

$$f^{-1}\left(\sum_{i=1}^m (c_i+h_i)e_i\right) = \sum_{i=1}^m (c_i+h_i)x_i \in O\left(\sum_{i=1}^m c_i x_i, \delta\right)\cap X_0 \subset U\cap X_0$$

proving that f^{-1} is continuous in the sense of topology.

Therefore X_0 is homeomorphic to $\mathbf{R}^m \subset \mathbf{R}^n$ and moreover $co\{x_1, \cdots, x_n\}$ is homeomorphic to $co\{e_1, \cdots, e_m, \sum_{i=1}^m a_{m+1,i}e_i, \cdots, \sum_{i=1}^m a_{n,i}e_i\}$. $\square$

By the above lemma we can show the following Takahashi's fixed point theorem in a vector lattice.

Theorem 1. *Let X be a Hausdorff Archimedean vector lattice with unit, Y a compact subset of X and Z a convex subset of Y. Suppose that a mapping f from Z into 2^Y satisfies*

(0) *$f^{-1}(y)$ is convex for any $y \in Y$,*

and there exists a mapping g from Z into 2^Y satisfying the following conditions:

(1) *$g(z)$ is a subset of $f(z)$ for any $z \in Z$;*

(2) *$g^{-1}(y)$ is non-empty for any $y \in Y$;*

(3) *$g(z)$ is an open subset of X for any $z \in Z$.*

Then there exists $z_0 \in Z$ such that $z_0 \in f(z_0)$.

Proof. By (2) it holds that $Y \subset \bigcup_{z\in Z} g(z)$. By (3) it holds that $g(z) \in \mathcal{O}_X$. Since Y is compact, there exists $z_1, \cdots, z_n \in Z$ such that $Y \subset \bigcup_{i=1}^n g(z_i)$. Since Y is normal, there exists a continuous function β_i in the sense of topology from Y into $[0,1]$ satisfying $\beta_i(y) = 0$ for any $y \in g(z_i)^C$ and $\sum_{i=1}^n \beta_i(y) = 1$. Let p be a mapping from Y into Z defined by $p(y) = \sum_{i=1}^n \beta_i(y)z_i$. Then p is continuous in the sense of topology. Since by (1) it holds that $g^{-1}(y) \subset f^{-1}(y)$, by (0) it holds that $p(y) \in f^{-1}(y)$. Let $Z_0 = co\{z_1, \cdots, z_n\}$. By Lemma 3 Z_0 is homeomorphic to a compact and convex subset K of $\mathbf{R}^n$. Put a mapping h from Z_0 into K as this homeomorphism. Then $h \circ p \circ h^{-1}$ is continuous in the sense of topology from K into K. Therefore by Brouwer's fixed point theorem there exists $x_0 \in K$ such that $h(p(h^{-1}(x_0))) = x_0$. Let $z_0 = h^{-1}(x_0)$. Then $p(z_0) = z_0$. Since $p(z_0) \in f^{-1}(z_0)$, it holds that $z_0 \in f^{-1}(z_0)$ proving that $z_0 \in f(z_0)$. $\square$

In the above theorem, putting $Z = Y$ and $g = f$, the following theorem is obtained. It is Fan-Browder's fixed point theorem in a vector lattice.

Theorem 2. *Let X be a Hausdorff Archimedean vector lattice with unit and Y a compact convex subset of X. Suppose that a mapping f from Y into 2^Y satisfies the following conditions:*

(1) *$f^{-1}(y)$ is non-empty and convex for any $y \in Y$;*

(2) *$f(y)$ is an open subset of X for any $y \in Y$.*

Then there exists $y_0 \in Y$ such that $y_0 \in f(y_0)$.

In the above theorem, changing from f to f^{-1}, the following theorem is obtained; see [7].

Theorem 3. *Let X be a Hausdorff Archimedean vector lattice with unit and Y a compact convex subset of X. Suppose that a mapping f from Y into 2^Y satisfies the following conditions:*

(1) *$f^{-1}(y)$ is an open subset of X for any $y \in Y$;*

(2) *$f(y)$ is non-empty and convex for any $y \in Y$.*

Then there exists $y_0 \in Y$ such that $y_0 \in f(y_0)$.

Moreover the following holds. For the sake of completeness, we show its proof.

Theorem 4. *Let X be a Hausdorff Archimedean vector lattice with unit, Y a compact convex subset of X and $A \subset Y \times Y$. Suppose that A satisfies the following conditions:*

(1) $\{x \mid (x,y) \in A\}$ *is closed for any* $y \in Y$;

(2) $\{y \mid (x,y) \notin A\}$ *is convex for any* $x \in Y$;

(3) $(x,x) \in A$ *for any* $x \in Y$.

Then there exists $x_0 \in Y$ *such that* $\{x_0\} \times Y \subset A$.

Proof. Assume that $\{x\} \times Y \not\subset A$ for any $x \in Y$. Then there exists $y \in Y$ such that $(x,y) \notin A$. Let $f(x) = \{y \mid (x,y) \notin A\}$. Then $f(x)$ is non-empty and by (2) it is convex. Moreover by (1) $f^{-1}(y) = \{x \mid (x,y) \notin A\} \in \mathcal{O}_X$. By Theorem 3 there exists $x_0 \in Y$ such that $x_0 \in f(x_0)$, that is, $(x_0, x_0) \notin A$. It is a contradiction. Therefore there exists $x_0 \in Y$ such that $\{x_0\} \times Y \subset A$. □

4 Schauder-Tychonoff's fixed point theorem

Let X be a vector lattice with unit and Y a vector lattice. Let $\mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ be the class of $\{v_e \mid e \in \mathcal{K}_X\}$ satisfying the following conditions:

(U1) $v_e \in Y$ with $v_e > 0$;

$(\mathrm{U2})^d$ $v_{e_1} \geq v_{e_2}$ if $e_1 \geq e_2$;

$(\mathrm{U3})^s$ For any $e \in \mathcal{K}_X$ there exists $\theta(e) \in \mathcal{K}_{\mathbf{R}}$ such that $v_{\theta(e)e} \leq \frac{1}{2} v_e$.

Let $x_0 \in Z \subset X$ and f a mapping from Z into Y. f is said to be continuous at x_0 if there exists $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists $\delta \in \mathcal{K}_{\mathbf{R}}$ such that for any $x \in Z$ if $|x - x_0| \leq \delta e$, then $|f(x) - f(x_0)| \leq v_e$. In particular if Y has an unit, then we consider often $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ satisfying the following condition instead of (U1):

$(\mathrm{U1})^u$ $v_e \in \mathcal{K}_Y$.

Example 1. We consider a sufficient condition such that there exists $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ satisfying the condition $(\mathrm{U1})^u$. Let X be an Archimedean vector lattice. Then there exists a positive homomorphism f from X into $\mathbf{R}$, that is, f satisfies the following conditions:

(H1) $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in X$ and for any $\alpha, \beta \in \mathbf{R}$;

(H2) $f(x) \geq 0$ for any $x \in X$ with $x \geq 0$.

Indeed it is shown as follows. By [8, Theorem IV.11.1] for any Archimedean vector lattice X there exists the completion $\hat{X}$ of X. By [8, Theorem V.4.2] for the complete vector lattice $\hat{X}$ there exists an extremally disconnected compact set Ω and a vector sublattice Y of $C_\infty(\Omega)$ such that $\hat{X}$ is isomorphic to Y, where

$$C_\infty(\Omega) = \left\{ f \;\middle|\; \begin{array}{l} f \text{ is continuous from } \Omega \text{ into } [-\infty, \infty] \text{ and} \\ f^{-1}(\{\pm\infty\}) \text{ is nowhere dense} \end{array} \right\}.$$

Therefore it may be assumed that X is a vector sublattice of $C_\infty(\Omega)$. Take $\omega \in \Omega$ arbitrary and let $f(x) = x(\omega)$ for any $x \in X$. Then f satisfies the conditions (H1) and (H2). Suppose that X satisfies that there exists a homomorphism f from X into $\mathbf{R}$ satisfying the following condition instead of (H2):

$(\mathrm{H2})^s$ $f(x) > 0$ for any $x \in X$ with $x > 0$.

Then for any $e_Y \in \mathcal{K}_Y$ $\{f(e)e_Y\}$ satisfies the conditions $(\mathrm{U1})^u(\mathrm{U2})^d(\mathrm{U3})^s$ clearly. Therefore if X is Archimedean and there exists a homomorphism from X into $\mathbf{R}$ satisfying the condition $(\mathrm{H2})^s$, then it may be assumed that every $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ satisfies the condition $(\mathrm{U1})^u$.

Let X and Y be vector lattices with unit, $Z \subset X$ and f a mapping from Z into Y. Suppose that there exists $P \subset Y$ satisfying the following conditions:

(P1) P is open and convex;

(P2) If $x \in P$ and $x \leq y$, then $y \in P$;

(P3) $0 \notin P$;

(P4) $\{x \mid x > 0\} \subset P$.

Let $\mathcal{P}_Y$ be the class of the above P's. f is said to be upper semi-continuous with respect to $P \in \mathcal{P}_Y$ if $\{x \mid y - f(x) \in P\} \in \mathcal{O}_X \cap Z$ for any $y \in Y$. f is said to be lower semi-continuous with respect to $P \in \mathcal{P}_Y$ if $\{x \mid f(x) - y \in P\} \in \mathcal{O}_X \cap Z$ for any $y \in Y$. f is said to be semi-continuous with respect to $P \in \mathcal{P}_Y$ if it is upper and lower semi-continuous with respect to $P \in \mathcal{P}_Y$.

Example 2. We consider of a sufficient condition to satisfy $\mathcal{P}_X \neq \emptyset$. Let X be an Archimedean vector lattice with unit. Suppose that there exists a homomorphism f from X into $\mathbf{R}$ satisfying the condition $(\mathrm{H2})^s$. Let $0 < \beta < 1$ and $\delta(x, e) = \frac{\beta f(x)}{f(e)}$ for any $x \in X$ with $x > 0$ and for any $e \in \mathcal{K}_X$. Put $P = \bigcup_{x \in X \text{ with } x > 0} int(O(x, \delta))$. Then P is open and $\{x \mid x > 0\} \subset P$.

Note that by the condition $(\mathrm{H2})^s$ for any $x_1, x_2 \in X$ with $x_1, x_2 > 0$ and $x_1 \neq x_2$, $\frac{x_1}{f(x_1)}$ and $\frac{x_2}{f(x_2)}$ are incomparable mutually. Therefore $x - \delta(x, e)e \not\leq 0$ for any $x \in X$ with $x > 0$ and for any $e \in \mathcal{K}_X$. Assume that $0 \in P$. Then there exists $x \in X$ with $x > 0$ and $e \in \mathcal{K}_X$ such that $0 \in [x - \delta(x, e)e, x + \delta(x, e)e]^e$. It is a contradiction. Therefore $0 \notin P$.

Note that $x \in int(A)$ if and only if there exists $\delta_x \in \Delta_X$ such that $O(x, \delta_x) \subset A$. Let $x \in P$ and $x \leq y$. Then there exists $z \in X$ with $z > 0$ and $\delta_x \in \Delta_X$ such that $O(x, \delta_x) \subset O(z, \delta)$. Let $\delta_y(u, e) = \delta_x(u - y + x, e)$. Since $\delta(x_2, e) \leq \delta(x_1 + x_2, e)$ for any $x_1, x_2 \in X$ with $x_1, x_2 > 0$, it holds that $x_1 + O(x_2, \delta) \subset O(x_1 + x_2, \delta)$. Therefore

$$O(y, \delta_y) = y - x + O(x, \delta_x) \subset y - x + O(z, \delta) \subset O(z + y - x, \delta),$$

that is, $y \in int(O(z + y - x, \delta)) \subset P$.

Let $x_0, x_1 \in P$ and $\alpha \in \mathbf{R}$ with $0 \leq \alpha \leq 1$. Then for $i = 0, 1$ there exists $y_i \in X$ with $y_i > 0$ and $\delta_i \in \Delta_X$ such that $O(x_i, \delta_i) \subset O(y_i, \delta)$. Let $\delta_\alpha(z, e) = (1 - \alpha)\delta_0(x_0, e) + \alpha\delta_1(x_1, e)$. Take $z \in O((1 - \alpha)x_0 + \alpha x_1, \delta_\alpha)$ arbitrary. Then there exists $e \in \mathcal{K}_X$ such that

$$\begin{aligned} z &\in [(1 - \alpha)x_0 + \alpha x_1 - \delta_\alpha((1 - \alpha)x_0 + \alpha x_1, e)e, \\ &\qquad (1 - \alpha)x_0 + \alpha x_1 + \delta_\alpha((1 - \alpha)x_0 + \alpha x_1, e)e]^e \\ &= (1 - \alpha)[x_0 - \delta_0(x_0, e)e, x_0 + \delta_0(x_0, e)e]^e + \alpha[x_1 - \delta_1(x_1, e)e, x_1 + \delta_1(x_1, e)e]^e. \end{aligned}$$

Since $\delta(\alpha x, e) = \alpha\delta(x, e)$ for any $x \in X$ with $x > 0$ and for any $\alpha \in \mathcal{K}_{\mathbf{R}}$, it holds that $O(\alpha x, \delta) = \alpha O(x, \delta)$. Since

$$\delta(z_0, e_0)e_0 + \delta(z_1, e_1)e_1 = \delta\left(z_0 + z_1, \frac{f(z_0)}{f(e_0)}e_0 + \frac{f(z_1)}{f(e_1)}e_1\right)\left(\frac{f(z_0)}{f(e_0)}e_0 + \frac{f(z_1)}{f(e_1)}e_1\right)$$

for any $z_0, z_1 \in X$ with $z_0, z_1 > 0$, it holds that $O(z_0, \delta) + O(z_1, \delta) \subset O(z_0 + z_1, \delta)$. Then

$$\begin{aligned} z &\in (1 - \alpha)O(x_0, \delta_0) + \alpha O(x_1, \delta_1) \\ &\subset (1 - \alpha)O(y_0, \delta) + \alpha O(y_1, \delta) = O((1 - \alpha)y_0, \delta) + O(\alpha y_1, \delta) \\ &\subset O((1 - \alpha)y_0 + \alpha y_1, \delta). \end{aligned}$$

Therefore $O((1 - \alpha)x_0 + \alpha x_1, \delta_\alpha) \subset O((1 - \alpha)y_0 + \alpha y_1, \delta)$, that is, $(1 - \alpha)x_0 + \alpha x_1 \in int(O((1 - \alpha)y_0 + \alpha y_1, \delta)) \subset P$.

Example 3. We consider of another simple sufficient condition to satisfy $\mathcal{P}_X \neq \emptyset$. Let X be a Hilbert lattice with unit, that is, X has an inner product $\langle \cdot, \cdot \rangle$ and for any $x, y \in X$ if $|x| \leq |y|$, then $\langle x, x \rangle \leq \langle y, y \rangle$. Then for any $e \in \mathcal{K}_X$ $P = \{x \mid \langle x, e \rangle > 0\}$ satisfies the conditions (P1)–(P4). Actually it is possible to show as follows.

It is clear that P is convex and $0 \notin P$.

Note that $\langle x, y \rangle \geq 0$ if $x, y \geq 0$. Actually since $|x - y| \leq x + y$ and $\langle |x - y|, |x - y| \rangle = \langle x - y, x - y \rangle$, it holds that $\langle x - y, x - y \rangle \leq \langle x + y, x + y \rangle$. Therefore it holds that $\langle x, y \rangle \geq 0$. Let $x \in P$ and $x \leq y$. Then $\langle y, e \rangle \geq \langle x, e \rangle > 0$ proving that $y \in P$.

Assume that there exists $x \in X$ with $x > 0$ such that $\langle x, e \rangle = 0$. Then since $\langle x + e, x + e \rangle = \langle x - e, x - e \rangle = \langle |x - e|, |x - e| \rangle$,

$$0 = \langle x + e + |x - e|, x + e - |x - e| \rangle = 4\langle x \vee e, x \wedge e \rangle \geq 4\langle x \wedge e, x \wedge e \rangle > 0.$$

It is a contradiction. Therefore $\{x \mid x > 0\} \subset P$.

For $x \in P$ and $e_1 \in \mathcal{K}_K$ putting $\delta < \frac{\langle x, e \rangle}{\langle e_1, e \rangle}$, then $\langle x - \delta e_1, e \rangle > 0$. Therefore P is open.

Lemma 4. *Let X be an Archimedean vector lattice with unit, Y a vector lattice with unit, $Z \subset X$ and f a mapping from Z into Y. Suppose that there exists a homomorphism from X into* $\mathbf{R}$ *satisfying the condition* (H2)^s *and that $\mathcal{P}_Y \neq \emptyset$. Then f is semi-continuous with respect to any $P \in \mathcal{P}_Y$ if it is continuous at any $x \in Z$.*

Proof. Take $y \in Y$ and $x_0 \in \{x \mid y - f(x) \in P\} \cap Z$ arbitrary. By the assumption there exists $\{v_e\} \in \mathcal{U}_Z^s(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists $\delta(e) \in \mathcal{K}_{\mathbf{R}}$ such that for any $x \in Z$ if $|x - x_0| \leq \delta(e)e$, then $|f(x) - f(x_0)| \leq v_e$. By the asumption it may be assumed that $v_e \in \mathcal{K}_Y$ for any $e \in \mathcal{K}_X$. Since P is open, there exists a natural number $n(e)$ such that $[y - f(x_0) - 2^{-n(e)}v_e, y - f(x_0) + 2^{-n(e)}v_e] \subset P$. If $|x - x_0| \leq \delta(\theta(e, n(e))e)\theta(e, n(e))e$, where $\theta(e, n) = \underbrace{\theta(\theta(\cdots\theta(\theta}_{n}(e)e)\cdots e)e)$, then $|f(x) - f(x_0)| \leq v_{\theta(e,n(e))e} \leq 2^{-n(e)}v_e$. Therefore $y - f(x) \in [y - f(x_0) - 2^{-n(e)}v_e, y - f(x_0) + 2^{-n(e)}v_e] \subset P$, that is, $[x_0 - \delta(\theta(e, n(e))e)\theta(e, n(e))e, x_0 + \delta(\theta(e, n(e))e)\theta(e, n(e))e] \subset \{x \mid y - f(x) \in P\} \cap Z$ proving that $\{x \mid y - f(x) \in P\} \in \mathcal{O}_X \cap Z$. Therefore f is upper semi-continuous with respect to P. Similarly it can be proved that f is lower semi-continuous with respect to P. □

Lemma 5. *Let X be an Archimedean vector lattice with unit, Y a vector lattice with unit, $x_0 \in Z \subset X$ and f a mapping from Z into Y. Suppose that there exists a homomorphism from X into* $\mathbf{R}$ *satisfying the condition* (H2)^s. *Then f is continuous at x_0 in the sense of topology if it is continuous at x_0.*

Proof. By the asumption there exists $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists $\delta(e) \in \mathcal{K}_{\mathbf{R}}$ such that for any $x \in Z$ if $|x - x_0| \leq \delta(e)e$, then $|f(x) - f(x_0)| \leq v_e$. By the asumption it may be assumed that $v_e \in \mathcal{K}_Y$ for any $e \in \mathcal{K}_X$. Let δ_Y be a gauge in Y. Take a natural number $n(e)$ such that $2^{-n(e)} < \delta_Y(f(x_0), v_e)$ and put $\delta_X(x, e) = \theta(e, n(e))\delta(\theta(e, n(e))e)$, where $\theta(e, n) = \underbrace{\theta(\theta(\cdots\theta(\theta}_{n}(e)e)\cdots e)e)$. Let $x \in O(x_0, \delta_X)$. There exists $e \in \mathcal{K}_X$ such that $x \in [x_0 - \delta_X(x_0, e)e, x_0 + \delta_X(x_0, e)e]^e$. Then

$$|f(x) - f(x_0)| \leq v_{\theta(e,n(e))e} \leq 2^{-n(e)}v_e < \delta_Y(f(x_0), v_e)v_e.$$

Therefore

$$f(x) \in [f(x_0) - \delta_Y(f(x_0), v_e)v_e, f(x_0) + \delta_Y(f(x_0), v_e)v_e]^{v_e} \subset O(f(x_0), \delta_Y)$$

proving that f is continuous at x_0 in the sense of topology. □

Theorem 5. *Let X be a Hausdorff Archimedean vector lattice with unit, Y a vector lattice with unit and Z a compact convex subset of X. Suppose that $\mathcal{P}_Y \neq \emptyset$ and that a mapping f from $Z \times Z$ into Y satisfies that there exists $P \in \mathcal{P}_Y$ such that*

(1) *$f(\cdot, x_2)$ is upper semi-continuous with respect to P for any $x_2 \in Z$;*

(2) *$f(x_1, \cdot)$ is convex for any $x_1 \in Z$;*

(3) *There exists $c \in Y$ such that $c - f(x, x) \notin P$ for any $x \in Z$.*

Then there exists $x_0 \in Z$ such that $c - f(x_0, x) \notin P$ for any $x \in Z$.

Proof. Let $A = \{(x_1, x_2) \mid c - f(x_1, x_2) \notin P\}$. By (1) $\{x_1 \mid (x_1, x_2) \in A\}$ is closed for any $x_2 \in Z$. By (3) $(x, x) \in A$ for any $x \in Z$. Let $z_1, z_2 \in \{x_2 \mid (x_1, x_2) \notin A\}$ and $0 \leq \alpha \leq 1$. By (2) and convexity of P

$$c - f(x_1, (1-\alpha)z_1 + \alpha z_2) \geq (1-\alpha)(c - f(x_1, z_1)) + \alpha(c - f(x_1, z_2)) \in P.$$

By (P2) $(1-\alpha)z_1 + \alpha z_2 \in \{x_2 \mid (x_1, x_2) \notin A\}$, that is, $\{x_2 \mid (x_1, x_2) \notin A\}$ is convex for any $x_1 \in Z$. By Theorem 4 there exists $x_0 \in Z$ such that $\{x_0\} \times Z \subset A$. Therefore $c - f(x_0, x) \notin P$ for any $x \in Z$. □

Theorem 6. *Let X be a Hausdorff Archimedean vector lattice with unit and Z a compact convex subset of X. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying the condition* (H2)^s *and that a mapping f from Z into X is continuous. Then it holds that* (1) *or* (2).

(1) *There exists $x_0 \in Z$ such that $f(x_0) = x_0$.*

(2) *There exists $x_0 \in Z$ such that $f(x_0) \neq x_0$ and $|x_0 - f(x_0)| - |x - f(x_0)| \notin P$ for any $P \in \mathcal{P}_X$ and for any $x \in Z$.*

Proof. Suppose that (1) is not satisfied. Then $f(x) \neq x$ for any $x \in Z$. Take $g(x_1, x_2) = |x_2 - f(x_1)| - |x_1 - f(x_1)|$. Then $g(\cdot, x_2)$ is continuous for any $x_2 \in Z$, $g(x_1, \cdot)$ is convex for any $x_1 \in Z$ and by (P3) $-g(x, x) = 0 \notin P$. By Lemma 4 and Theorem 5 there exists $x_0 \in Z$ such that $-g(x_0, x) = |x_0 - f(x_0)| - |x - f(x_0)| \notin P$ for any $x \in Z$. □

Theorem 7. *Let X be a Hausdorff Archimedean vector lattice with unit and Z a compact convex subset of X. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying the condition* (H2)^s *and that a mapping f from Z into X is continuous. Then there exists $x_0 \in Z$ such that $f(x_0) = x_0$.*

Proof. Assume that (2) in Theorem 6 holds. Then there exists $x_0 \in Z$ such that $f(x_0) \neq x_0$ and $|x_0 - f(x_0)| - |x - f(x_0)| \notin P$ for any $x \in Z$. Since $f(x_0) \neq x_0$, by (P4) $|x_0 - f(x_0)| \in P$. Take $x = f(x_0)$. Then $|x_0 - f(x_0)| \notin P$. It is a contradiction. Therefore there exists $x_0 \in Z$ such that $f(x_0) = x_0$. □

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References

[1] R. Cristescu, *Topological Vector Spaces*, Noordhoff International Publishing, Leyden, 1977. Zbl 0345.46001

[2] T. Kawasaki, *Denjoy integral and Henstock-Kurzweil integral in vector lattices, I, II*, Czechoslovak Math. J. **59** (2009), 381–399, 401–417.

[3] T. Kawasaki, M. Toyoda, and T. Watanabe, *Fixed point theorem for set-valued mapping in a Riesz space*, Memoirs of the Faculty of Engineering, Tamagawa University **44** (2009), 81–85 (in Japanese).

[4] ———, *Takahashi's and Fan-Browder's fixed point theorems in a vector lattice*, J. Nonlinear Convex Anal., to appear.

[5] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces*, North-Holland, Amsterdam, 1971. Zbl 0231.46014

[6] W. Takahashi, *Fixed point, minimax, and Hahn-Banach theorems*, Proc. of Symposia in Pure Math. **45** (1986), 419–427. Zbl 0636.47048

[7] ———, *Nonlinear Functional Analysis. Fixed Points Theory and its Applications*, Yokohama Publishers, Yokohama, 2000. Zbl 0997.47002

[8] B. Z. Vulikh, *Introduction to the Theory of Partially Orderd Spaces*, Wolters-Noordhoff, Groningen, 1967. Zbl 0186.44601